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NOTE

Analysis of a Fourth-Order Compact Scheme for Convection–Diffusion[1]

1. INTRODUCTION

In 1984 Gupta *et al.* [3] introduced a compact fourth-order finite-difference convection–diffusion operator with some very favorable properties. In particular, this scheme does not seem to suffer excessively from spurious oscillatory behavior, and it converges with standard methods such as Gauss Seidel or SOR (hence, multigrid) regardless of the diffusion [7]. This scheme has been rederived, developed (including some variations), and applied in both convection–diffusion and Navier–Stokes equations by several authors [1, 2, 4–6]. Accurate solutions to high Reynolds-number flow problems at relatively coarse resolutions have been reported (e.g., [1, 2]). These solutions were often compared to those obtained by lower order discretizations, such as second-order central differences and first-order upstream discretizations. The latter, it was stated, achieved far less accurate results due to the artificial viscosity, which the compact scheme did not include [3].

We show here that, while the compact scheme indeed does not suffer from a cross-stream artificial viscosity (as does the first-order upstream scheme when the characteristic direction is not aligned with the grid), it does include a streamwise artificial viscosity that is inversely proportional to the natural viscosity. This term is not always benign.

2. THE EQUATION AND ITS DISCRETIZATION

To be consistent with [3] we write the convection–diffusion equation with Dirichlet boundary conditions in two dimensions as

$$\begin{aligned} u_{xx} + u_{yy} + \overline{p}u_x + \overline{q}u_y &= \overline{f}, \quad (x, y) \in \Omega, \\ u(x, y) &= g \quad (x, y) \in \partial\Omega. \end{aligned} \tag{1}$$

Here, u is the solution and $\overline{p}$, $\overline{q}$, and $\overline{f}$ are generally functions of x and y. Here we shall assume constant $\overline{p}$ and $\overline{q}$ in order to greatly simplify the discussion. As we are interested in the convection-dominated case, which would imply very large $\overline{p}$ and $\overline{q}$ in (1), we rescale this equation as

$$p = \varepsilon\overline{p}, \quad q = \varepsilon\overline{q}, \quad f = \varepsilon\overline{f},$$

where $\varepsilon = 1/\max(|p|, |q|)$. Multiplying (1) through by ε we obtain

$$\varepsilon(u_{xx} + u_{yy}) + pu_x + qu_y = f \tag{2}$$

in Ω, with the same boundary condition as in (1). Since the convection operator is a (scaled) derivative in the characteristic direction, which we denote by ξ, we denote this operator by

$$D_\xi = p\partial_x + q\partial_y.$$

Suppose that we discretize (2) on a uniform grid of meshsize h. For constant p and q we can write the discretized problem using the compact fourth-order scheme of [3] as

$$\left[\varepsilon\Delta^h + D_\xi^h + \frac{h^2}{12\varepsilon} D_{\xi\xi}^h\right] u^h = \left[I^h + \frac{h^2}{12}\tilde{\Delta}^h + \frac{h^2}{12\varepsilon}\tilde{D}_\xi^h\right] f^h, \tag{3}$$

where h superscripts denote discrete values and operators. Here, I^h is the grid-h identity. The rest of the discrete operators are defined by the following stencils. We also write the differential operators they approximate and corresponding truncation errors obtained by Taylor series expansions:

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$$\Delta^h = \frac{1}{6h^2}\begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} = \Delta + \frac{h^2}{12}\Delta^2 + O(h^4),$$

$$D^h_\xi = \frac{1}{12h}\begin{bmatrix} -p+q & 4q & p+q \\ -4p & 0 & 4p \\ -p-q & -4q & p-q \end{bmatrix}$$

$$= D_\xi + \frac{h^2}{6}\Delta D_\xi + O(h^4),$$

$$D^h_{\xi\xi} = \frac{1}{2h^2}\begin{bmatrix} -pq & 2q^2 & pq \\ 2p^2 & -4(p^2+q^2) & 2p^2 \\ pq & 2q^2 & -pq \end{bmatrix} \qquad (4)$$

$$= D_{\xi\xi} + O(h^2),$$

$$\tilde{\Delta}^h = \frac{1}{h^2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \Delta + O(h^2),$$

$$\tilde{D}^h_\xi = \frac{1}{2h}\begin{bmatrix} 0 & q & 0 \\ -p & 0 & p \\ 0 & -q & 0 \end{bmatrix} = D_\xi + O(h^2).$$

Here, Δ is the Laplacian and $D_{\xi\xi}$ is the square of the convection operator:

$$D_{\xi\xi} = p^2\partial_{xx} + 2pq\partial_{xy} + q^2\partial_{yy}.$$

Now, substitution of the differential operators and truncation terms of (4) into (3) produces, after some manipulation utilizing the commutativity of Δ and D_ξ,

$$\left(1 + \frac{h^2}{12}\Delta + \frac{h^2}{12\varepsilon}D_\xi\right)(\varepsilon\,\Delta u^h + D_\xi u^h - f^h) = O((1+\varepsilon^{-1})h^4), \qquad (5)$$

where we have assumed differentiable extensions of u^h and f^h to Ω for convenience of notation.

Equation (5) allows us to analyze the scheme. We assume smooth u^h and f^h. Clearly the discretization is consistent since, for a fixed ε, u^h satisfies the differential equation when h tends to zero. Formally, the right-hand side of (5) is $O(h^4)$, consistent with the scheme's known fourth-order accuracy. But the size of this truncation term also depends on ε. Thus, for any fixed ε we expect the accuracy of a smooth solution to improve by a factor 16 when we halve the meshsize, say. But when $\varepsilon \approx h$ (cell-Reynolds number approximately equal to one), the right-hand side is actually $O(h^3)$, and when $\varepsilon \approx h^2$ it is $O(h^2)$. The scheme thus becomes less accurate in the convection-dominated case.

Consider, moreover, the limit $\varepsilon/h^2 \ll 1$. The dominant terms in (5) then satisfy

$$D_\xi(\varepsilon\,\Delta u^h + D_\xi u^h - f^h) = O(h^2). \qquad (6)$$

Now the discretization no longer approximates the convection equation but rather its streamwise derivative (with $O(h^2)$ relative truncation error). As a result the discretization becomes insensitive to f^h that is constant along the characteristic direction, though the original equation is not. Furthermore, the main term in the approximated equation is now a streamwise second derivative instead of a first derivative. Hence, both "inflow" and "outflow" boundary conditions strongly affect the solution throughout the domain, instead of there being an $O(\varepsilon)$ boundary layer. In effect, the operator has become fully viscous in the streamwise direction.

In intermediate regimes the approximated equation is some combination of the convection equation and its streamwise derivative. The effect of the latter becomes noticeable when the cell Reynolds number is approximately one, since then the streamwise viscosity term is roughly equal to the natural viscosity. If we further reduce the viscosity ε the solution will actually become more and more viscous in the streamwise direction. We expect the boundary layer to be thinnest when the sum of the natural viscosity and the artificial streamwise viscosity is minimal, i.e., $\varepsilon = 12^{-1/2}h$.

TABLE I

Computed Boundary-Layer Decay Rates Compared with the Exact Rate

Exact	4th Order	Upstream
(ε/h)	($\tilde{\varepsilon}/h$)	
2.0000	2.0002	2.4623
1.0000	1.0015	1.4426
0.4000	0.4300	0.7982
0.3000	0.3804	0.6820
0.2887	0.3797	0.6684
0.2500	0.3899	0.6213
0.2000	0.4427	0.5581
0.0400	2.0835	0.3069
0.0040	16.912	0.1810

3. NUMERICAL EXAMPLES

In our examples $\Omega = (0, 1) \times (0, 1)$, and we use a grid of 41 by 41 points, including the boundary, so that $h = 0.025$. We compare solutions obtained with the fourth-order compact scheme to those produced by a standard first-order upstream discretization (in fact downstream, because the viscosity here was chosen with a positive sign for consistency with [3], but this is moot).

EXAMPLE 1. We choose the simple problem $p = 1$, $q = f = 0$, with $u = 1$ at $x = 0$, and $u = 0$ at $x = 1$. The top and bottom boundary conditions are chosen to be consistent with the 1D solution, i.e.,

$$u = \frac{\exp(-x/\varepsilon) - \exp(-1/\varepsilon)}{1 - \exp(-1/\varepsilon)}.$$

The small-ε solution is very close to zero throughout the domain except in an $O(\varepsilon)$ boundary layer near $x = 0$, where it decays exponentially from its boundary value of 1. Hence, we test the accuracy of the computed results by fitting the function $\exp(-x/\tilde{\varepsilon})$ to the calculated solutions at $y = 0.5$ from $x = 0$ to $x = 0.25$, and comparing $\tilde{\varepsilon}$ to ε. The results appear in Table 1 for several values of ε/h. For $\varepsilon/h \geq 1$ the high-order scheme is very accurate. But just as predicted, the least viscous solution occurs at $\varepsilon/h = 12^{-1/2}$. For smaller values the boundary layer thickens monotonically as predicted by our analysis. For example, at $\varepsilon = 1/1000$ (eighth row) the solution is close to that of $\varepsilon = 1/20$ (first row), because the streamwise "artificial viscosity" is $h^2/12\varepsilon = 1/19.2$).

The low-order solution's boundary-layer thickness is not as good as that of the high-order scheme when ε/h is moderate, but it is reduced monotonically as ε is decreased. Of course the main weakness of the upstream discretization is concealed in this example due to the perfect alignment of the characteristics with the grid and the absence of cross-stream variation.

EXAMPLE 2. On the same domain let $p = q = 1$, and let

$$f = \begin{cases} -1 & y > x, \\ 0, & y = x, \\ 1, & y < x, \end{cases}$$

with $u \equiv 0$ on $\partial\Omega$. Note that f is constant along characteristics. Here the solution in the inviscid limit decays linearly in the left-upper triangle and grows linearly the right-lower

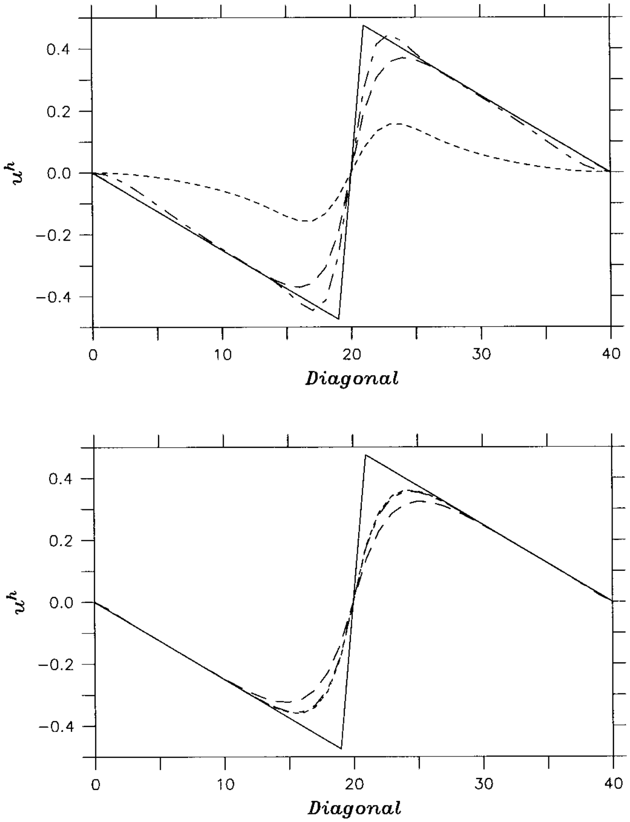

FIG. 1. Diagonal cross-section of u^h obtained with the high-order scheme (top) and upstream scheme (bottom). The solid line shows the exact inviscid solution. Long dashes correspond to $\varepsilon = 0.01$, alternating dashes to $\varepsilon = 0.001$, and short dashes to $\varepsilon = 0.0001$.

triangle. In Fig. 1 we plot the solutions along the cross-section $x = 1 - y$ from the upper left to the lower right corner of the domain. The inviscid "sawtooth" exact solution (restricted to the present grid) is shown by the solid lines in Fig. 1. Here, it is not *a priori* obvious what the optimal ε will be for the high-order scheme, because the cross-stream viscosity plays an important role. It turns out that the least damped solution is obtained at about $\varepsilon = 0.001$. When ε is reduced further, the insensitivity of the streamwise second derivative to f that is independent on the streamwise coordinate leads to the expected deterioration in the solution. Note also that there is some overshoot in the small-ε results, although it does not seem to be very severe.

For the upstream scheme this is a worst-case example because of the maximal nonalignment and the strong adverse effect of cross-stream viscosity. The results with this scheme improve monotonically as ε is reduced and with no oscillatory behavior, as expected. But they are not as good as the best results obtained with the high-order scheme. Here, the use of a so-called "narrow" upstream scheme would improve the results significantly, and more sophisticated higher-order schemes with limiters would do still better.

4. DISCUSSION AND CONCLUSIONS

For large cell Reynolds numbers our analysis and simple tests show that the compact fourth-order convection–diffusion scheme may yield spurious behavior in at least two cases: (a) Due to boundary information propagating nonphysically along the characteristics. (b) Due to a forcing function f, or some component of it, that is constant along characteristics. Suitable caution is therefore recommended when using this scheme in such situations even for solutions that are very smooth almost everywhere. It would be useful if the deficiencies reported here could be overcome by employing alternative boundary conditions, but we could not find a reasonable way to do this.

Our findings do not contradict reports of accurate incompressible Navier Stokes solutions to the driven cavity problem, for example. In this problem all streamlines form closed curves, and the characteristics never cross a boundary, so our first example does not apply. Also, the vorticity equation has zero right-hand side, so the second example does not apply either. Indeed it may well be that the deficiencies reported here do not appear in any important problem of purely recirculating flow, since such problems are ill-posed in the limit of vanishing viscosity unless the integral of the right-hand side function along each closed characteristic vanishes. This precludes a nonzero component of f that is constant along the characteristic. However, this matter requires further investigation.

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